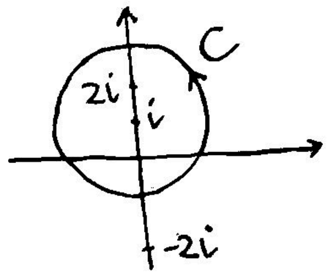
Tutorial Note   Mar. 12

1. Evaluate the integral of $g(z)$ around the circle $|z-i|=2$ oriented positively, when

(a) $g(z) = \dfrac{1}{z^2+4}$, (b) $g(z) = \dfrac{1}{(z^2+4)^2}$



Solution.

(a) Recall the Cauchy integral formula:

" If $f(z)$ is holomorphic everywhere inside a positively oriented simply closed curve C, then $\forall z_0$ inside C, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad "$$

We decompose $g(z)$ into $g(z) = \dfrac{f(z)}{z-2i}$, where $f(z) = \dfrac{1}{z+2i}$

One can check directly that $f(z)$ is holomorphic everywhere inside C.

Therefore, $\displaystyle\int_C g(z)\,dz = \int_C \frac{f(z)}{z-2i}\,dz = 2\pi i\, f(2i) = \frac{\pi}{2}$

(b) Recall the Cauchy integral formula for derivatives:

" Under the same hypothesis of Cauchy integral formula, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad "$$

We decompose $g(z)$ into $g(z) = \dfrac{f(z)}{(z-2i)^2}$, where $f(z) = \dfrac{1}{(z+2i)^2}$

One can check directly that $f(z)$ is holomorphic everywhere inside C

Therefore, $\displaystyle\int_C g(z)\,dz = \int_C \frac{f(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{1!}\, f'(2i) = 2\pi i\left(-\frac{2}{(z+2i)^3}\right)\Bigg|_{z=2i}$

$= \dfrac{\pi}{16}$

$\square$

2. Suppose $f(z)$ is an entire function, and there is a non-empty disk such that $f(z)$ does not attain any values in the disk. Prove that $f(z)$ is constant.

Pf. We first recall the Liouville Theorem.

Liouville Thm. If f is entire and bounded in the complex plane, then f is a constant.

This problem is an application of the Liouville theorem.

Step 1. We may assume without loss of generality that

(*) $f(z)$ cannot attain any values in $B_0(\delta) = \{z, |z| \le \delta\}$.

Indeed, we may consider $f(z) - z_0$ if the disk is centred at z_0.

Step 2. It follows from (*) that $|f(z)| \ge \delta \quad \forall z \in \mathbb{C}$

Thus, we can define $g(z) = \frac{1}{f(z)}$, which is well-defined for $\forall z \in \mathbb{C}$ and it is an entire function with $|g(z)| \le \delta^{-1}$.

Step 3. We apply the Liouville Theorem to yield $g(z) \equiv \text{const}$, which implies $f(z) \equiv \text{const}$. $\square$

Remark. The Range of a non-constant entire function is dense in $\mathbb{C}$.

3. Suppose that $f(z)$ is entire and $u(x,y) = \mathrm{Re}(f(z)) \le u_0 \in \mathbb{R}$ for $\forall (x,y) \in \mathbb{R}^2$. Prove that $f \equiv \text{const}$.

Pf. Consider $g(z) = \exp(f(z))$, then $g(z)$ is entire and

$$|g(z)| = |\exp(u(z) + i\,v(z))| = |\exp(u(z))| \le \exp(u_0)$$

Thus, it follows from the Liouville theorem that $g \equiv \text{const}$, and

Thus $f \equiv \text{const}$. $\square$

Remark. The same conclusion holds if "$u \le u_0$" is replaced by

(i) "$u \ge u_0$" : we may consider $g(z) = \exp(-f(z))$

(ii) "$v \le v_0$" : we may consider $g(z) = \exp(-if(z))$ $[v = \mathrm{Im}(f)]$

(iii) "$v \ge v_0$" : we may consider $g(z) = \exp(if(z))$.

4. Assume that f is continuous in a closed bounded region R, and f is analytic and non-constant in the interior of R. Prove that both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ cannot achieve their maximum and minimum values in the interior of R, i.e, they can only be achieved on the boundary of R.

Pf. We first recall the Maximum Modulus Principle :

"Thm. If f is non-constant and holomorphic in R, then $|f(z)|$ has no global maximum inside R, where R is a closed & bounded set."

We apply maximum modulus principle to $g(z) = \exp(f(z))$.

Notice that $|g(z)| = \exp(\mathrm{Re}(f(z)))$, and its global max cannot be achieved in the interior of R. Therefore, the global max of $\mathrm{Re}(f(z))$ can only be achieved on the boundary of R, but not in the interior of R.

Similarly, we can prove the remaining three propositions by considering

(i) $g(z) = \exp(-f(z))$ for min of $\mathrm{Re}(f(z))$

(ii) $g(z) = \exp(-if(z))$ for max of $\mathrm{Im}(f(z))$

(iii) $g(z) = \exp(if(z))$ for min of $\mathrm{Im}(f(z))$ $\square$

5. Find the Maclaurin series expansion for

(a) $f(z) = \dfrac{z}{z^4+4}$ at $z=0$;

(b) $f(z) = \cos z$ at $z = \dfrac{\pi}{2}$.

Sol. (a) We recall the Maclaurin series for $g(z) = \dfrac{1}{1-z}$:

$$g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z|<1.$$

Notice that $f(z) = \dfrac{z}{4}\, g\left(-\dfrac{z^4}{4}\right)$, we would thus have

$$f(z) = \sum_{n=0}^{\infty} \frac{z}{4}\left(-\frac{z^4}{4}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1}, \quad \text{for } \left|\frac{z^4}{4}\right| < 1$$

i.e. $|z| < \sqrt{2}$.

(b) We notice that $\cos z = -\sin\left(z - \frac{\pi}{2}\right)$,

and $\sin z = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$, for $|z| < \infty$.

Thus,
$$\cos z = -\sin\left(z-\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty} (-1)^n \frac{\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}, \quad \text{for } \left|z-\frac{\pi}{2}\right| < \infty.$$

□